SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS

INTRODUCTION

- Let there be m first degree equations with n unknowns $x_1, x_2, x_3, \ldots\ldots, x_n$ as

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1j} x_j + \ldots\ldots + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2j} x_j + \ldots\ldots + a_{2n} x_n = b_2$

....

....

$a_{i1} x_1 + a_{i2} x_2 + \ldots\ldots + a_{ij} x_j + \ldots\ldots + a_{in} x_n = b_i$

....

$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots + a_{mj} x_j + \ldots\ldots + a_{mn} x_n = b_m$

If all the b_i's vanishes then the given system of equations is ***homogeneous***

otherwise ***non-homogeneous system***

METHODS TO OBTAINING THE SOLUTIONS

If given system of equations is AX = B

Where A=[aij]m×n ,

X = [x1 , x2, x3 ,......, xm]’ 1×m

and

B = [b1,b2,b3,.......,bm]’ 1×m

Then the following methods are used:

1. If B=0,detA = 0,then there exist infinite number of solutions besides trivial solution X=0.
2. If B=0 and detA =! 0, then system has only unique trivial solution,X=0.
3. If B=! 0 and detA =! 0,then system has a unique solution.

GUASS ELIMINATION METHOD

- Let the system of linear equations be

$a11x1 + a21x2 + a13x3 = b1$ .....(1)

$a21x1 + a22x2 + a23x3 = b2$ .....(2)

$a31x1 + a32x2 + a33x3 = b3$ .....(3)

First we eliminate x1 from equations (1) and (2) and then from (1) and (3).

From these two obtained equations, eliminate x2 obtaining the value of x3.

Then by back substitutions,we get x2 and x1.

For the elimination of x1 from (1) and (2) multiply (1) by r1, where

r= a21/a11

and subtract the equation so obtained from equation (2) i.e. $(a_{21} - r_1 a_{12})x_2 + (a_{23} - r_1 a_{13})x_3 = b_2 - r_1 b_1$ .....(4)

Similarly, for eliminating x_1 from equations (1) and (3), we multiply equation (1) by r_2

where

$$r_2 = a_{31}/a_{11}$$

and subtract the equation so obtained from (3) to get

$(a_{32} - r_2 a_{12})x_2 + (a_{33} - r_2 a_{13})x_3 = b_3 - r_2 b_1$ ......(5)

where

$a_{22} - r_1 a_{12} = p_{22}$, $b_2 - r_1 b_1 = c_2$, $a_{23} - r_1 a_{13} = p_{23}$

$a_{32} - r_2 a_{12} = p_{32}$, $a_{33} - r_2 a_{13} = p_{33}$, $b_3 - r_2 b_1 = c_3$

Now equations (4) and (5) cam be written as

$p_{22} x_2 + p_{23} x_3 = c_2$ ......(6)

$p_{32} x_2 + p_{33} x_3 = c_3$ ......(7)

- Now to eliminate $x2$ from these two equations , multiply eq (6) by $r3$, where

$$r3 = p32/p22$$

and subtract the equation so obtained from (7) , to get

$$(p33 - r3\ p23)\ x3 = c3 - r3\ c2$$

$$\Rightarrow \quad x3 = (c3 - r3\ c2) / (p33 - r3\ p23)$$

Now put the value of $x3$ in (6) or (7) , $x2$ can be obtained.

Then substitute the values of $x2$ and $x3$ in (1) or (2) or (3) ,

We can get the value of $x1$.

Example

$2x + y + 4z = 12$ ....(1)

$4x + 11y - z = 33$ ....(2)

$8x - 3y + 2z = 20$ ....(3)

First eliminate x from (1) and (2)

using $r1 = a21/a11 = 4/2 = 2$

Multiply (1) by 2 and subtracting from (2) , we get

$(4x + 11y - z) - 2(2x + y + 4z) = 33 - 2(12)$

$\Rightarrow$ $9x - 9z = 9$

$\Rightarrow$ $y - z = 1$ ....(4)

Now eliminate x from (1) and (3) ,

Using $r2 = a31/a11 = 8/2 = 4$

Multiply (1) by 4 and subtracting from (3) , we get

$(8x - 3y + 2z) - 4(2x + y + 4z) = 20 - 4(12)$

$\Rightarrow$ $-7y - 14z = -28$

$\Rightarrow$ $y + 2z = 4$ ....(5)

Now eliminate y from (4) and (5)

Using r3 = coefficient of y in (5) / coefficient of y in (4)

= 1/1 = 1

Subtracting (4) from (5) , we get

$(y + 2z) - (y - z) = 4-1$

$\Rightarrow$ $3z = 3$

$\Rightarrow$ $z = 1$

Put $z = 1$ in (5) , we get $y = 2$

And put $z = 1$ and $y = 2$ in (1) , we get $x = 3$

Hence the solution is $x = 3$, $y = 2$, $z = 1$.

GUASS-JORDAN METHOD

- This method is a modification of previous method.In this method , elimination

of unknowns is performed in such a way that system of equations reduces to a diagonal matrix form.

Consider the equations

$$a1x + b1y + c1z = d1 \quad ....(1)$$

$$a2x + b2y + c2z = d2 \quad ....(2)$$

$$a3x + b3y + c3z = d3 \quad ....(3)$$

Assume that a1 =! 0

Elimination of x from equations (2) and (3) :-

Operating eqn(2) – a2/a1 × eqn(1) and eqn(3) – a3/a1 × eqn(1) ,

the reduced system of equations is

$$a1x + b1y + c1z = d1 \qquad \text{........(4)}$$

$$b2'y + c2'z = d2' \qquad \text{........(5)}$$

$$b3'y + c3'z = d3' \qquad \text{........(6)}$$

Elimination of y from equations (4) and (6) :-

Operating eq(4) – b1/b2' × eq(5) and eq(6) – b3'/b2' × eq(5)

The reduced system of equations is

$$a1''x + c1''z = d1'' \qquad \text{........(7)}$$

$$b2'y + c2'z = d2' \qquad \text{........(8)}$$

$$c3''z = d3'' \qquad \text{........(9)}$$

Elimination of z from equations (7) and (8) :-

Operating eq(7) – c1''/c3'' × eq(9) amd eq(8) – c2'/c3'' × eq(9)

The reduced system of equations is

$$a1'''x = d1'''$$

$$b2''y = d2''$$

$$c3''z = d3''$$

Hence the required solution is $x = d1'''/a1'''$, $y = d2''/b2''$, $z = d3''/c3''$

EXAMPLE

$x + y + z = 9$ ....(1)

$2x - 3y + 4z = 13$ .....(2)

$3x + 4y + 5z = 40$ .....(3)

To eliminate x from eqns (2) and (3) :-

Operating eq(2) – 2× eq(1) and eq(3) – 3× eq(1) , the reduced system of equations is

$x + y + z = 9$ ....(4)

$-5y + 2z = -5$ ....(5)

$y + 2z = 13$ ....(6)

To eliminate y from eqns (4) and (6):-

Operating eq (4) + 1/5 × eq(5) and eq(6) + 1/5× eq(5),the reduced system of equations is

$x + 7/5\, z = 8$ ....(7)

$-5y + 2z = -5$ ....(8)

$12/5\, z = 12$ ....(9)

To eliminate z from eq(7) and eq(8):-

Operating eq (7) – 7/12 × eq(9) and eq(8) – 5/6× eq(9),the reduced system of equations is

$$x = 1$$

$$-5y = -15$$

$$12/5\, z = 12$$

Hence the required solution is

$$x = 1$$

$$y = -3$$

and $z = 5$

TRIANGULARIZATION METHOD
OR
LU DECOMPOSITION METHOD

Consider a system of n linear equations in n unknowns.

Im triangularization method, the square matrix A (coefficient matrix) is factorized or decomposed into the product of two matrices L and U, where L is a lower triangular matrix and U is an upper triangular matrix.

Thus, we write A as A = LU, where

$$L = \begin{bmatrix} L11 & 0 & 0 & \cdots & 0 \\ L21 & L22 & 0 & \cdots & 0 \\ \\ \\ Ln1 & Ln2 & Ln3 & \cdots & Lnn \end{bmatrix} \text{ and } U = \begin{bmatrix} u11 & u12 & u13 & \cdots & u1n \\ 0 & u22 & u23 & \cdots & u2n \\ \\ \\ 0 & 0 & 0 & \cdots & unn \end{bmatrix}$$

Now, multiply the matrices L and U and compare the corresponding elements of resulting matrix with elements of matrix A, we ontain the equation of form aij = Lij uij + L2j u2j + ... + Lin unj ; i = 1,2,...,n ; j = 1,2,....,n,

Where Lij = 0 for j>1 and uij = 0 for i > j.

To obtain unique solution, we can either choose Lii = 1 or uii = 1 ; i = 1,2,....,n.

METHOD 1. *DOOLITTLE'S METHOD* : In this method, we take Lii = 1 i.e., the coefficient matrix A is decomposed into product of a lower triangular matrix L with diagonal elements as unity and an upper triangular matrix U.This method is called Doolittle's method.

First express coefficient matrix as A = LU .....(1)

and obtain the elements of L and U by equating elements of matrix LU with corresponding elments of matrix A.

After obtaining all the elements of L and U ,proceed as follows:-

Write AX = B as

(LU)X = B or L(UX) = B ....(2)

Let UX = Z ....(3)

then (2) becomes LZ = B ....(4)

where Z = [z1 z2 z3]'

Put values of L and B in (4) to find values of Z(z1,z2,z3) and then put value of Z and U in (3) to get the values of X(x1,x2,x3) which is the required solution.

METHOD 2. ***CROUT'S TRIANGULARIZATION METHOD*** **:-**

This method is same as Doolittle's method except that diagonal elements of U is taken as unity instead of that of L.

REMARKS :- 1. Triangularization method is applicable if all the principal minors of matrix A are non-zero.

Alternatively, this method fails if any of the diagonal elements Lii or uii is zero.

2. If this method fails, then we should rearrange the guven equations in such a way, so that all the principal minors are non-zero.

EXAMPLE

SOLVE by LU method :

$x + y + z = 1$

$4x + 3y + z = 6$

$3x + 5y + 3z = 4$

SOLUTION: Let given system of equations is AX = B ......(1)

Let A = LU, where L is lower triangular matrix with diagonal elements as unity and U is upper triangular matrix.

i.e.

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ L_{21}u_{11} & L_{21}u_{12} + u_{22} & L_{21}u_{13} + u_{23} \\ L_{31}u_{11} & L_{31}u_{12} + L_{32}u_{22} & L_{31}u_{13} + L_{32}u_{23} + u_{33} \end{bmatrix} \quad ......(2)$$

Compare the elements on both sides row and column wise amd solve for the values to obtain the elements of L and U .

We get $u_{11} = 1$, $u_{12} = 1$, $u_{13} = 1$

$L_{21} = 4$, $L_{31} = 3$, $u_{22} = -1$

$u_{23} = -5$, $L_{32} = -2$, $u_{33} = -10$

Thus

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

Putting A = LU in (1), we get LUX = B ......(3)

Let UX = Z where Z = [z1,z2,z3]' ....(4)

Hence eqn (3) becomes LZ = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z1 \\ z2 \\ z3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

Multiplying and comparing the corresponding elements, we have

z1 = 1 ; z2 = 2 ; z3 = 5

i.e. Z = [1 2 5]'

Putting the values of U and Z in (4), we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Solving we get x = 1, y= ½ , z = -1/2 which is the required solution.

CROUT'S METHOD

Let given system of equations is

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Which can be written as AX = B ....(1)

Augmented matrix [A:B] is

$$\begin{matrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{matrix}$$

From this we derive another matrix [A':B'] called the **Derived or Auxiliary matrix as follows:**

1. The elements of first column of new matrix are same as that of [A:B]

i.e., $a_{11}' = a_{11}$, $a_{21}' = a_{21}$, $a_{31}' = a_{31}$

2. For elements of first row(except a11'),divide the elements of first row of [A:B] by a11

i.e. $a12' = a12/a11$, $a13' = a13/a11$, $b1' = b1/a11'$

3. For elements of a22',a32' of 2nd column:

$a22' = a22 - a21' a12'$

$a32' = a32 - a31' a12'$

4. For elements a23',b2' of 2nd row:

$a23' = (a23 - a21' a13')/a22'$

$b2' = (b2 - a21' b1')/a22'$

5. For elements of 3rd column :

$a33' = a33 - a31' a13' - a32' a23'$

6. For last element b3' of 3rd row:

$b3' = (b3 - a31' b1' - a32' b2')/a33'$

7. The vues of **x1,x2,x3 are found as**

x3 = b3' from 3rd row

x2 = b2' – a23' x3 frm 2nd row

x1 = b1' – a11' x2- a12' x3 from 1st row

EXAMPLE

- **Solve by Crout's method:**

$x + y + 2z = 7$

$3x + 2y + 4z = 13$

$4x + 3y + 2z = 8$

Solution. Augmented matrix is

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 2 & : & 7 \\ 3 & 2 & 4 & : & 13 \\ 4 & 3 & 2 & : & 8 \end{bmatrix}$$

The elements of derived matrix are found as:

1. **Elements of 1st column:**

a11' = a11, a21' = a21, a31'= a31

Hence a11' = 1, a21' = 3 , a31' = 4.

2. Elements of 1st row(except a11') :

a12' = a12/a11, a13' = a13/a11 , b1' = b1/a11

Hence a12' = 1, a13' = 2, b1' = 7

3. Elements of 2nd column (except a12') :

a22' = a22 – a21' a12' = 2 – 3×1 = -1

a32' = a32 – a31' a12' = 3 – 4×1 = -1

4. Elements of 2nd row i.e., a23', b2' :

a23' = (a23- a21' a13')/a22' = (4-3×2)/(-1) = 2

b2' = (b2- a21' b1')/a22' = (13-3×7)/(-1) = 8

5. Element a33' of 3rd column :

a33' = a33 – a31' a13' – a32' a23'

= 2 – 4×2 – (-1)×2 = -4

6. Elements b3' of 3rd row :

b3' = (b3 – a31' b1' – a32' b2')/a33'

= (8 – 4.7 – (-1).8) /(-4)

= (8 – 28 + 8)/(-4)

= 3

Thus, the Derived matrix

$$[A' : B'] = \begin{matrix} 1 & 1 & 2 & : & 7 \\ 3 & -1 & 2 & : & 8 \\ 4 & -1 & -4 & : & 3 \end{matrix}$$

Therefore solution of given system of equations is

$z = b_3' = 3,$

$y = b_2' - a_{23}' z$

$= 8 - 2.3$

$= 2$

$x = b_1' - a_{12}' y - a_{13}' z$

$= 7 - 1.2 - 2.3$

$= -1$

Hence the required solution is $x = -1$, $y = 2$, $z = 3$

ITERATIVE METHODS
1.Jacobi's Method

- This iteration method is applicable only when eqch equation of rthe system contains one coefficient much larger than the other in that equation and the larger coefficients in different equations correspond to the different unknowns.

Let the system of thrse non homogeneous linear equations in three variables be

$$a11\ x1 + a12\ x2 + a13\ x3 = b1$$

$$a21\ x1 + a22\ x2 + a23x3 = b2 \qquad .....(1)$$

$$a31\ x1 + a32\ x2 + a33\ x3 = b3$$

Let us assume that a11,a22,a33 be the largest coefficient in 1^{st},2^{nd} and 3^{rd} equation of (1).

Then on solving forx1,x2,x3 the system if eq(1) can written as

$x1 = (b1 - a12x2 - a13\ x3)/a11$

$x2 = (b2 - a21\ x1 - a23\ x3)/a22$ ......(2)

$x3 = (b3 - a31\ x1 - a32\ x2)/a33$

For the **first iteration :** Let the initial approximation be x1 = x10, x2 = x20, x3 = x30

Substituting these values on right sides of equations (2) , we have

$x11 = (b1 - a12\ x20 - a13\ x30)/a11$

$x21 = (b2 - a21\ x10 - a23\ x30)/a22$ .....(3)

$x31 = (b3 - a31\ x10 - a32\ x20)/ a33$

For the second **iteration :** substitute values of x11,x21, x31 on right sides of equations (2),

We get

$x12 = (b1 - a12\ x21 - a13\ x31)/a11$

$x22 = (b2 - a21\ x11 - a23\ x31)/a22$

$x32 = (b3 - a31\ x11 - a32\ x21)/a33$

The process is repeated, till the values of x1,x2,x3 are obtained to the desired degree of accuracy.

Example: $20x + y - 2z = 17$
$3x + 20y - z = -18$ and $2x - 3y + 20z = 25$

Solution: consider given system as eqn (1)

Here the coefficients of x,y and z in the 1st,2nd and 3rd equations of (1) are larger than the coefficients of other variables in the respective equations.Hence we can solve these equations by Jacobi method.

Solving each equation for unknown having larger coefficient,we have

$x = (17 - y + 2z)/20$

$y = (-18 - 3x + z)/20$

$z = (25 - 2x + 3y)/20.$ ....(2)

For the 1st iteration :

Let initial approximation be $x = 0$, $y = 0$, $z = 0$. substitute these values in (2) we get x1 = 17/20 = 0.85

y1 = -18/20 = 0.9

z1 = 25/20 = 1.25

For the 2nd iteration : Put values of x1,y1, and z1 in (2), we get

x2 = (17 - (-0.9)+2(-1.25))/20 = 1.02

y2 = (-18-3(0.85)+(-1.25))/20 = -0.965

z2 = (25 – 2(0.85)+3(0.9))/20 = 1.03

For the 3rd iteration : Put values if x2,y2 and z2 in (2), we get

x3 = (17 –(-0.965)+2(1.03))/20 = 1.00125

y3 = (-18-3(1.02)+1.03))/20 = -1.0015

z3 = (25-2(1.02)+3(-0.965))/20 = 1.00325

For the 4th iteration : put values if x3,y3 and z3 in (2) , we get

x4 = (17-(-1.0015)+2(1.00325))/20 = 1.0004

y4 = (-18-3(-1.00125)+1.00325)/20 = -1.000025

z4 = (25 -2(1.00125)+3(1.0015))/20 = 0.9965

For the 5th iteration : put values ifx4,y4 and z4 in (2),we get

x5 = (17-(-1.000025)+2(0.9965))/20 = 0.999966

y5 = (-18-3(-1.0004)+0.9965) = -1.000078

z5 = (25-2(1.0004)+3(-1.000025))/20 = 0.999956

The values of the forth and fifth iterations are almost same.Hence we can stop the iteration process and the required solution is

x = 1 , y = -1 , z = 1.

GUASS-SEIDEL METHOD

This method is an improvement over Jacobi's Method.It is different from previous method in computing the iteration values by using previous iterated values at every step.

Let the system of three non-homogeneous linear equations in theee variables be

$$a_{11} x_1 + a_{12} x_2 + a_{12} x_3 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3 \quad ....(1)$$

Which can be written as

$$x_1 = (b_1 - a_{12} x_2 - a_{12} x_3)/a_{11}$$

$$x_2 = (b_2 - a_{21} x_1 - a_{23} x_3)/a_{22}$$

$$x_3 = (b_3 - a_{31} x_1 - a_{32} x_2)/a_{33} \quad ....(2)$$

Here, we start with initial approximation $x_2 = x_{20}$, $x_3 = x_{30}$(these values x_{20} and x_{30} may each be taken as zero). On substituting these values of x_2 and x_3 in the first of equations (2), we have

$$x_{11} = (b_1 - a_{12} x_{20} - a_{13} x_{30})/a_{11}$$

Now on putting $x = x_{11}$ and $x_3 = x_{30}$ in the second of equations (2), we have

$$x_{21} = (b_2 - a_{21}x_{11} - a_{23}x_{30})/a_{22}$$

Next, substituting $x_1 = x_{11}$ and $x_2 = x_{21}$ in the third of equations (2), we have

$$x_{31} = (b_3 - a_{31}x_{11} - a_{32}x_{21})/a_{33}$$

And so on i.e., as soon as a new approximation for an unknown is found, it is immediately used in the next step. This process of iteration is repeated till the values of x_1, x_2, x_3 are obtained to the desired degree of accuracy.

EXAMPLE. APPLY GUASS-SEIDEL METHOD TO SOLVE

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

Solution. Solving each equation for the unknown having the larger coefficients, we have

$$x = (17 - y + 2z)/20 \quad ......(1)$$

$y = (18 - 3x + z)/20$ .....(2)

$z = (25 - 2x + 3y)/20$ .....(3)

For the 1st iteration : Let the initial approximation be y = 0 and z = 0. On substituting these values in (1), we have

$x1 = (17 - 0 + 2\times0)/20 = 0.8500$

Now putting x = x1, z = 0 in (2), we get

$y1 = (-18 - 3\times0.8500 + 0)/20 = -1.0275$

Putting x = x1 and y = y1 in (3),we get

$z1 = (25 - 2\times 0.8500 + 3(-1.0275))/20 = 1.0109$

For the 2nd iteration :

Putting y = y1 and z = z1 in (1), we get

$x2 = (17 - (-1.0275) + 2(1.0109))/20 = 1.0025$

Putting x = x2 and z = z1 in (2), we get

$y2 = (-18-8(1.0025+1.0109)/20 = -0.9998$

Putting x = x2 and y = y2 in (3), we get

$z2 = (25-2(1.0025)-3(-0.9998))/20 = 0.9998$

For the 3rd iteration :

Putting y = y2 and z = z2 in (1), we get

$x3 = (17-(-0.9998)+2(0.9998))/20 = 1.0000$

Putting x = x3 and z = z2 in (2), we get

$y3 = (-18-3(1.0000)+0.9998)/20 = -1.0000$

Putting x = x3 and y = y3 in (3), we get

$z3 =(25-2(1.0000)+3(-1.0000))/20 = 1.0000$

Since the values obtained in 2nd and 3rd iterations are very close, we can stop the process and hence the solution is x = 1, y = -1, z = 1.

Relaxation Method

Let the system of equations be

$a11\ x1 + a12\ x2 + a13\ x3 = b1$

$a21\ x1 + a22\ x2 + a23\ x3 = b2$

$a31\ x1 + a32\ x2 + a33\ x3 = b3$ ......(1)

We define the residuals R1,R2,R3 by the relations

$R1 = b1 - a11x1 - a12x2 - a13x3$

$R2 = b2 - a21x1 - a22x2 - a23x3$

$R3 = b3 - a31x1 - a32x2 - a33x3$ .....(2)

Let the initial approximation be x1 = 0,x2 = 0 and x3 = 0.

Substituting these values in (2), initial residuals are R1 = b1, R2 = b2, R3 = b3.

Now these residuals are reduced step by bstep by giving increments to the variables. For this purpose we construct the following operation table :

	dR1	dR2	dR3
dx1 = 1	-a11	-a21	-a31
dx2 = 1	-a21	-a22	-a32
dx3 = 1	-a13	-a23	-a33

where dx_1 stand for increment in x_1 and dR_1, dR_2, dR_3 stands for corresponding increments in R_1, R_2 and R_3 respectively etc.

Now we reduce the numerically largest residual to almost zero. To reduce a particular residual, the value of corresponding variable is changed. For example, to reduce $R_{11} = b_1$ by a, we increase x by a/a_{11}.

We repeat the same procedure till all the residuals are reduced to almost zero.Then the sum of increments in x_1, x_2, x_3 are the values of x_1, x_2, x_3 respectively.

Hence the required solution is obtained.

REMARK : Relaxation method is generally used when the diagonal elements of the coefficient matrix dominate the other coefficients in the corresponding row i.e., if in eqn (1), we have

$|a_{11}| >= |a_{12}| + |a_{13}|$, $|a_{22}| >= |a_{21}| + |a_{23}|$

and $|a_{33}| >= |a_{31}| + |a_{32}|$

EXAMPLE.SOLVE BY RELAXATION METHOD:

12x1 + x2 + x3 = 31

2x1 + 8x2 - x3 = 24

3x1 + 4x2 + 10x3 = 58

Solution : we observe that the diagonal elements of the coefficient matrix dominate the other coefficients i.e., |12| >= |1| + |1| , |8| >= |2| + |-1| and |10| >= |3| + |4|, so relaxation method can be applied.

The residuals are given by

R1 = 31 – 12x1 – x2- x3

R2 = 24 – 2x1 – 8x2 + x3

R3 = 58 – 3x1 – 4x2 – 10x3

The operation table is

	dR1	dR2	dR3
dx1 = 1	-12	-2	-3
dx2 = 1	-1	-8	-4
dx3 = 1	-1	1	-10

Let the initial approximation be x1 = x2 = x3 = 0. Then the initial residuals are R1 = 31,R2 = 24 and R3 = 58.

Steps of making relaxation table :

1. **First** of all we choose the largest residual numerically.Here R3 = 58 is numerically the largest residual. Now we should add a suitable multiple of third row of the operation table to the first row of the relaxation table, so that R3 = 58 is almost reduced to zero.
2. **We** take dx3 = 5. Multiplying the third row in the operation table by 5 and adding it to the initial residuals, we get the new residuals as **R1 = 26, R2 = 29, R3 = 8.**
3. Among these R2 = 29 is numerically the largest residual. To reduce R2, we take dx2 = 3. Multiplying the second row of operation table by 3 and adding it to the residuals obtained in the step 2, we get the new residuals as **R1=23, R2=5,R3=-4.**
4. Aming these R1 = 23 is numerically the largest residual. To reduce R1, we take dx1 = 1. Multiplying the first row of operation table by 1 and adding it to the residuals obtained in the step 3, we get the new residuals as **R1=11, R2= 3,R3=-7.**
5. Among these R1 = 11 is numerically the largest residual. To reduce R1, we take dx1 = 1. Multiplying the first row of operation table by 1 and adding it to the residuals obtained in the step 4, we get the new residuals as **R1=0, R2=1,R3= -10.**

6. Now, among these R3 = -10 is numerically the largest residual. To reduce R3, we take dx3 = -1. Multiplying the third row of operation table by -1 and adding it to the residuals obtained in the step 5, we get the new residuals as

R1 = 0, R2 = 0 R3 = 0.

Thus the relaxation table is

	R1	R2	R3
x1 = x2 = x3 = 0	31	24	**58**
dx3 = 5	26	**29**	8
dx2 = 3	**23**	5	-4
dx1 = 1	**11**	3	-7
dx1 = 1	-1	1	**-10**
dx3 = -1	0	0	0

Now, x1 = sum of increments in x1 = £ dx1 = 1 + 1 = 2

x2 = sum of increments in x2 = £ dx2 = 3

x3 = sum of increments in x3 = £ dx3 = 5 – 1 = 4

Hence, the required solution is x1 = 2, x2 = 3, x3 = 4.

ASSIGNMENT

Solve by Guass Elimination Method :

1. $2x + 3y - z = 5$

$4x + 4y - 3z = 3$

$2x - 3y + 2z = 2$

2. $5x - y - 2z = 142.2$

$x - 3y - z = -30$

$2x - y - 3z = 5$

Solve by Guass-Jordan Method :

1. $2x - 3y + z = -1$

$x + 4y + 5z = 25$

$3x - 4y + z = 2$

2. $10x + 3y + z = 67$

$2x + 5y + 2z = 10$

$3x - 2y + 5z = 40$

Solve by triangular method.

1. $2x + y + z = 2$

$x + 3y + 2z = 2$

$3x + y + 2z = 2$

2. $2x1 + x2 + 2x3 = 2$

$x1 + 5x2 + 3x3 = 4$

$x1 + x2 - x3 = 0$

Solve by Crout's method :

1. $2x - 2y - 4z = 1$
$2x + 3y + 2z = 9$
$-x + y + z = \frac{1}{2}$

2. $x + y + z = 2$
$2x + 3y - 2z = -4$
$2x + y - 2z = 1$

Solve by Jacobi's method :

1. $5x + 2y + z = 12$
$x + 4y + 2z = 15$
$x + 2y + 5z = 20$

2. $5x - y + z = 10$
$2x + 4y = 12$
$x + y + 5z = -1$

Solve by Guass-Jordan method :

1. $10x + 2y + z = 9$
$2x + 20y - 2z = -44$
$-2x + 3y + 10z = 22$

2. $10x + y + z = 6$
$x + 10y + z = 6$
$x + y + 10z = 6$

Solve by Relaxation Method :

1. $9x - y + 2z = 9$
$x + 10y - 2z = 15$
$2x - 2y - 13z = -17$

2. $9x - 3y - 4z = 100$
$x - 7y + 3z = -80$
$2x + 3y - 5z = -60$